Choice Under Certainty

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decision theory

- Decision theory is about making choices
  - Normative aspect: what "rational" people should do
  - Descriptive aspect: what people do do
- Not surprisingly, it's been studied by economists, psychologist, and philosophers
- More recently, computer scientists have looked at it too
  - How should we design robots that make reasonable decisions
  - What about software agents acting on our behalf
    - Agents bidding for you on eBay
    - Managed health care
  - Algorithmic issues in decision making
- This course will focus on normative aspects, informed by a computer science perspective

axiomatic decision theory

Standard (mathematical) approach to decision theory:

- Give axioms characterizing reasonable decisions
- Ones that any “rational” person should accept

- Then show show that these axioms characterize a particular approach to decision making
- For example, we will discuss Savage’s axioms that characterize maximizing expected utility

- An issue that arises frequently:
- How to represent a decision problem

- Let X be a set of alternatives
- Typical elements of X are denoted by $x, y, z \ldots$
- For each pair $x, y \in X$ we ask a subject whether he prefers x over y, y over x, or neither
- Notation: $x \succ y$ means the subject strictly prefers x over y
- The relation $\succ$ is a binary relation on X

- **Example:** $X = \{a, b, c\}$, $b \succ a$, $a \succ c$, and $b \succ c$
- What if the individual also said $a \succ b$?

- What are some (arguably) reasonable properties for a preference order?
- A binary relation $\succ$ on X satisfies
  - **Completeness** if $x \not\succ y$ implies $y \succ x$
  - **Asymmetry** if $x \succ y$ implies $y \not\succ x$
  - **Acyclicity** if $x_1 \succ x_2 \succ \ldots \succ x_n$ implies $x_1 \neq x_n$
  - **Transitivity** if $[x \succ y$ and $y \succ z]$ implies $x \succ z$
  - **Negative transitivity** if $[x \not\succ y$ and $y \not\succ z]$ implies $x \not\succ z$

- **Example:** $X = \{a, b, c\}$, $a \succ b$, $b \succ c$, $c \succ a$
  - Are such cyclic preferences reasonable?
  - How would this person choose an alternative from X?

Definition: A binary relation $\succ$ is a strict preference relation if it is asymmetric and negatively transitive

- **Example:** $X = \{a, b, c\}$, and $\succ$ is a strict preference relation with $b \succ a$, and $a \succ c$.
- What can we tell about b vs. c?
  - Asymmetry implies $c \not\succ a$ and $a \not\succ b$
  - If $b \not\succ c$, then NT would imply $b \not\succ a$ ▼
  - Hence, it must be that $b \succ c$
  - Asymmetry then implies $c \not\succ b$

Proposition: The binary relation $\succ$ is NT iff $x \succ z$ implies that for all $y \in X$, $y \succ z$ or $x \succ y$

Proposition: If $\succ$ is NT then $[x \not\succ y$ and $y \not\succ x]$ imply that for all z, $[z \succ x$ iff $z \succ y]$ and $[x \succ z$ iff $y \succ z]$

- **Example:** Let $X = \{1, 2, 3, \ldots\}$ and suppose that 1 is not ranked by $\succ$ relative to any other alternative
- If $\succ$ satisfies NT, then there are no alternatives which can be ranked

Definition: Given a relation $\succ$ on X

1. x is weakly preferred to y, $x \succcurlyeq y$, iff $y \not\succ x$
2. x is indifferent to y, $x \sim y$, iff $y \not\succ x$ and $x \not\succ y$

Proposition: Given a relation $\succ$ on X

1. $\succ$ is asymmetric if and only if $\nprec$ is complete
2. $\succ$ is negatively transitive if and only if $\nprec$ is transitive

- Proof of $\Rightarrow$
  1. Asymmetry implies we cannot have both $x \succ y$ and $y \succ x$

     Hence, at least one of $x \nprec y$ and $y \nprec x$ holds
  2. If $x \nprec y \nprec z$, then $z \not\succ y \not\succ x$

     Therefore, by NT, $z \not\succ z$, i.e., $x \nprec z$
- Proof of $\Leftarrow$ will be on homework 1

transitivity

- Why do we care about transitivity?

- People's preferences sometimes fail transitivity
- However, if this is pointed out, most people think they should change their preferences

- **Example:** $X = \{a, b, c\}$, $a \succ b$, $b \succ c$, $c \succ a$
  - No undominated alternative in X
  - $a \succcurlyeq b$ and $b \succcurlyeq c$, but $a \not\succcurlyeq c$
  - Susceptible to money pumps
    - Starting from a, pay a penny to switch from a to c
    - Then pay a penny to switch from c to b
    - Then pay a penny to switch from b to a
    - ...

Proposition: If $\succ$ is a strict preference relation, then it is transitive and acyclic

- Proof for transitivity:
  - Suppose $x \succ y \succ z$, we need to show $x \succ z$
  - Asymmetry implies $z \not\succ y$
  - If $x \not\succ z$, NT would imply $x \not\succ y$ ▼
  - Hence we must have $x \succ z$
- Proof for acyclicity:
  - Suppose towards a contradiction $x_1 \succ x_2 \succ \ldots \succ x_n$ and $x_1 = x_n$
  - Transitivity would imply $x_1 \succ x_n = x_1$, which contradicts asymmetry ▼

- **Example:** $X = \{a, b, c\}$, $a \succ b$, $b \succ c$, no other strict preference rankings
  - The relation $\succ$ is clearly acyclic (and asymmetric)
  - It is not transitive because $a \succ b \succ c$ but $a \not\succ c$
  - Can you come up with a relation that is acyclic and complete but not transitive?

- **Example:** $X = \{a, b, c\}$, $a \succ c$, no other strict preference rankings
  - The relation $\succ$ is clearly transitive and asymmetric
  - It is not NT because $a \not\succ b \not\succ c$ but $a \succ c$
  - Can you come up with an example of a relation that is NT but not transitive?
  - What if we also require that $x \succ y$ or $y \succ x$ whenever $x \neq y$?

- $X = \{1, 2, 3, \ldots\}$, number of grains of sugar in my coffee
- I do not like sugar, but I cannot distinguish less than one teaspoon
  - If $|x - y| \leq 10{,}000$, then $x \not\succ y$ and $y \not\succ x$
  - If $x - y > 10{,}000$, then $y \succ x$
- The relation $\succ$ is transitive and asymmetric
  - If $x \succ y$, then $x < y$ and thus $y \not\succ x$
  - If $x \succ y \succ z$, then $z - x = (z - y) + (y - x) > 20{,}000$, and thus $x \succ z$
- The induced indifference relation $\sim$ is not transitive
  - $1 \sim 2 \sim 3 \sim \ldots \sim 9{,}999 \sim 10{,}000 \sim 10{,}001$, but $1 \not\sim 10{,}001$

- Hard to measure preferences beyond introspection and surveys
- Choices can be observed to some extent

- Maintained assumption: X is finite
- Let $P(X)$ denote the set of all non-empty subsets of X
- Typical elements of $P(X)$ are denoted by A, B, ...

> **Definition:** A choice function is a function $c : P(X) \to P(X)$ such that $c(A) \subseteq A$ for every $A \in P(X)$

- $X = \{a, b, c\}$
- By definition $c(\{a\}) = \{a\}$, $c(\{b\}) = \{b\}$ and $c(\{c\}) = \{c\}$

A	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	$\{x, y, z\}$
$c(A)$	$\{x\}$	$\{x, z\}$	$\{z\}$	??

- $X = \{a, b, c\}$
- By definition $c(\{a\}) = \{a\}$, $c(\{b\}) = \{b\}$ and $c(\{c\}) = \{c\}$

A	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	$\{x, y, z\}$
$c(A)$	$\{x\}$	$\{x, z\}$	$\{z\}$	$\{x, z\}$
$c'(A)$	$\{x\}$	$\{x, z\}$	$\{y\}$	$\{y, z\}$
$c''(A)$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y\}$
$c'''(A)$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	$\{x, y, z\}$

to some extent

- Knowledge of a choice function requires observing
  - Choices from different alternative sets ceteris paribus
  - Choices from all elements of $P(X)$
  - All possibly chosen alternatives from each element of $P(X)$

- How would a decision maker with preferences $\succ$ make choices?
  - Presumably it would never choose dominated alternatives
  - Any undominated alternative would be acceptable

Definition: The choice function induced by a preference $\succ$ on X is the function $c(\,\cdot\,,\succ) : P(X) \to P(X)$ given by

$$c(A,\succ) = \left\{\, x \in A \;\middle|\; \text{for all } y \in A,\ y \not\succ x \,\right\}$$

- **Example:** $X = \{a, b, c\}$, $a \succ b$, $b \succ c$, and $a \succ c$
  - $c(\{a, b\},\succ) = \{a\}$, $c(\{a, c\},\succ) = \{a\}$, $c(\{a, b, c\},\succ) = \{a\}$
  - $c(\{b, c\},\succ) = \{b\}$

Proposition: If $\succ$ is acyclic and X is finite, then $c(\,\cdot\,,\succ)$ is a choice function

- Proof that $c(A,\succ) \neq \emptyset$ for all $A \in P(X)$
  - Let $m < \infty$ be the number of alternatives in A
  - Suppose towards a contradiction that $c(A,\succ) = \emptyset$ and fix some $x_1 \in A$
  - There would exist $x_2 \in A$ such that $x_2 \succ x_1$ and $x_2 \neq x_1$ (why?)
  - There would exist $x_3 \in A$ such that $x_3 \succ x_2 \succ x_1$ and $x_3 \neq x_1, x_2$

    $\vdots$
  - There would exist $x_{m+1} \in A$ such that $x_{m+1} \succ \ldots \succ x_3 \succ x_2 \succ x_1$ and $x_{m+1} \neq x_1, x_2, \ldots x_m$ $\blacktriangledown$
- If X is finite and $c(\,\cdot\,,\succ)$ is a choice function, then $\succ$ is acyclic

- What are the implications of preferences in terms of choices?
- If c is an arbitrary choice function, need there be a preference $\succ$ such that $c(\,\cdot\,) = c(\,\cdot\,, \succ)$?

- **Example:** $X = \{a, b, c\}$, $c(\{a, b, c\}) = c(\{a\}) = \{a\}$ and $c(\{a, b\}) = \{b\}$
  - $c(\{a\}) = \{a\}$ requires $a \not\succ a$
  - Then $c(\{a, b\}) = \{b\}$ requires $b \succ a$
  - But then $c(\{a, b, c\}) = \{a\}$ is not possible
  - Is there a context where these choices would be reasonable?

Where do we stand?

If $\succ$ is a preference order (negatively transitive + asymmetric) on a set X, then we can partition the elements of X into "indifference classes" $X_1, \ldots, X_k$ such that "$X_1 \succ X_2 \succ \ldots \succ X_k$"

- more precisely, if $x, y \in X_i$, then $x \sim y$ (i.e., $x \not\succ y$ and $y \not\succ x$)
- if $x \in X_i$ and $y \in X_j$ where $i < j$, then $x \succ y$

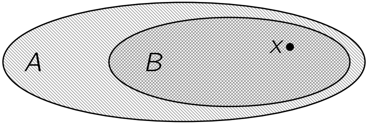
Now suppose that $\succ_{po}$ is such that $x \succ_{po} z$, and y is incomparable to x and z ("po" = "partial order")

- then $\succ_{po}$ can't be a preference order ($x \sim_{po} y$, $y \sim_{po} z$, but $x \succ_{po} z$)
- But $\succ_{po}$ is acyclic
- So there is a choice function c induced by $\succ_{po}$:
  $c(\{x, z\}) = x$, $c(\{x, y\}) = \{x, y\}$, $c(\{y, z\}) = \{y, z\}$,
  $c(\{x, y, z\}) = \{x, y\}$

What properties does a choice function c induced by a preference order have?

- We'll consider two properties of a choice function c that are necessary and sufficient for c to be induced by a preference order
  - every choice function induced by a preference order has these properties.
  - if a choice function c has these properties, then $c(\cdot) = c(\cdot, \succ)$ for some preference order $\succ$

Sen's α



Sen's axiom α: If $x \in B \subseteq A$ and $x \in c(A)$, then $x \in c(B)$

If the smartest student in the U.S. is from Cornell,
then he/she is also the smartest student at Cornell.

Proposition: For any preference relation $\succ$, the induced choice function $c(\,\cdot\,, \succ)$ satisfies axiom α

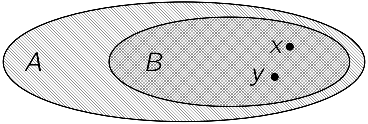
- Proof:
  - Suppose $x \in B \subseteq A \subseteq X$ and $x \in c(A, \succ)$, but $x \notin c(B, \succ)$
  - $x \notin c(B, \succ)$ implies that $y \succ x$ for some $y \in B$
  - Since $y \in A$, then $x \notin c(A, \succ)$ ▼

Actually, even if $\succ$ is acyclic, $c(\,\cdot\,, \succ)$ satisfies axiom α

- The same proof applies without change!

So if $\succ$ is a preference order, there must be additional properties that $c(\,\cdot\,, \succ)$ satisfies.

Sen's β



Sen's axiom β: If $B \subseteq A$, $x, y \in c(B)$, and $y \in c(A)$, then $x \in c(A)$

If the smartest student in the U.S. is from Cornell, then all the smartest students at Cornell are among the smartest students in the U.S.

Proposition: For any preference relation $\succ$, the induced choice function $c(\,\cdot\,,\succ)$ satisfies axiom β

- Proof:
  - Suppose $x, y \in B \subseteq A \subseteq X$, $x \in c(B,\succ)$, and $y \in c(A,\succ)$
  - $x, y \in c(B,\succ)$ implies that $y \not\succ x$
  - $y \in c(A,\succ)$ implies that $z \not\succ y$ for all $z \in A$
  - NT then implies that $z \not\succ x$ for all $z \in A$
  - Hence, $x \in c(A,\succ)$

$c = c(\,\cdot\,,\succ_{po})$ does *not* satisfy axiom β:

- $z \in c(\{y,z\})$, $y \in c(\{x,y,z\})$, but $z \notin c(\{x,y,z\})$

- If $c = c(\cdot, \succ)$ and $\succ$ is a preference order, then it satisfies axioms α and β
- Are there any more requirements? No

Theorem:

(a) If $\succ$ is a preference relation on a finite set X, then $c(\cdot) = c(\cdot, \succ)$ is a choice function satisfying axioms α and β

(b) If c is a choice function on X satisfying axioms α and β, then $c(\cdot) = c(\cdot, \succ)$ for a unique preference relation $\succ$

We've already shown part (a).

- But note that we need X to be finite. Why?
  - To ensure that $\succ$ is acyclic, and thus to ensure that $c(A, \succ)$ is nonempty if A is nonempty.

We need to work a little to prove part (b) . . .

- Given a choice function c satisfying axioms α and β, we want to find a preference relation $\succ$ such that $c(\cdot) = c(\cdot, \succ)$
  - Given x and y, how can we figure out if $x \succ y$ or $y \succ x$?
  - Look at $c(\{x, y\})$!

Definition: Given a choice function c, x is revealed preferred to y if $x \neq y$ and $c(\{x, y\}) = \{x\}$

- Let $\succ$ denote the revealed preferences from c, we need to show
  1. $c(A) \subseteq c(A, \succ)$ for all $A \in P(X)$
  2. $c(A, \succ) \subseteq c(A)$ for all $A \in P(X)$
  3. $\succ$ is asymmetric
  4. $\succ$ is negatively transitive

proof

1. $c(A) \subseteq c(A, \succ)$ for all $A \in P(X)$
   - Let $x \in c(A)$
   - By Axiom α, $x \in c(\{x, y\})$ for all $y \in A$
   - Hence, $y \not\succ x$ for all $y \in A$

2. $c(A, \succ) \subseteq c(A)$ for all $A \in P(X)$
   - Let $x \in c(A, \succ)$ and take any $y \in c(A)$
   - Axiom α implies $y \in c(\{x, y\})$
   - Since $x \in c(A, \succ)$, $y \not\succ x$ and thus $c(\{x, y\}) \neq \{y\}$
   - Therefore $c(\{x, y\}) = \{x, y\}$
   - Axiom β then implies $x \in c(A)$

3. $\succ$ is asymmetric

   – $x \not\succ x$ by construction

   – If $x \succ y$ then $c(\{x, y\}) = \{x\}$, so $c(\{x, y\}) \neq \{y\}$, hence $y \not\succ x$

4. $\succ$ is negatively transitive

   – Suppose that $z \not\succ y \not\succ x$; we want to show that $z \not\succ x$; that is, $x \in c(\{x, z\})$

   – By Axiom α, it suffices to show that $x \in c(\{x, y, z\})$

   – If $y \in c(\{x, y, z\})$

     Since $y \not\succ x$, we must have $x \in c(\{x, y\})$

     Axiom β then implies $x \in c(\{x, y, z\})$

   – If $z \in c(\{x, y, z\})$ then an analogous argument implies $y \in c(\{x, y, z\})$ and thus, $x \in c(\{x, y, z\})$

   – If $y \notin c(\{x, y, z\})$ and $z \notin c(\{x, y, z\})$ then $c(\{x, y, z\}) = \{x\}$